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Number lines, but not area models, support children's accuracy and conceptual models of fraction division

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ABSTRACT

The Common Core State Standards in Mathematics recommends that children should use visual models to represent fraction operations, such as fraction division. However, there is little experimental research on which visual models are the most effective for helping children to accurately solve and conceptualize these operations. In the current study, 123 fifth and sixth grade students solved fraction division problems in one of four visual model conditions: number lines, circular area models, rectangular area models, or no visual model at all. Children who solved the problems accompanied by a number line were more accurate and showed evidence of consistently producing sound conceptual models across the majority of problems than did children who completed problems with either area model or no visual model at all. These findings are particularly striking given that children have experienced partitioning area models into equal shares as early as first grade, thus circles and rectangles were likely familiar to children. The number line advantage may stem from the fact that they afford the ability to represent both operand magnitudes in relation to one another and relative to a common endpoint. Future work should investigate the optimal order that instructors should introduce various visual models to promote children's representational fluency across number lines and area models.

1. Introduction

Reasoning about fraction operations is a critical aspect in the development of children's deep understanding of mathematics. The National Mathematics Advisory Panel (NMAP, 2008) considers understanding fractions to be foundational for algebra (p. xviii). Empirically, children's understanding of fractions predicts later mathematics achievement and success in algebra (e.g., Bailey, Hoard, Nugent, & Geary, 2012; Siegler et al., 2012). However, despite the importance of children's understanding of fraction operations, this facet of early and middle mathematics is notoriously difficult for children (e.g., Mack, 1990, 1995, 2001; Siegler, Thompson, & Schneider, 2011) and adults (e.g., Ball, 1990; Luo, Lo, & Leu, 2011; Ma, 1999). Even though fraction learning begins early in first grade, many students continue to struggle to accurately represent and perform fraction operations (Lortie-Forgues, Tian, & Siegler, 2015; Sidney & Alibali, 2015, 2017; Siegler & Pyke, 2013; Siegler et al., 2011).

One common way of supporting children's understanding of challenging fraction concepts is by using visual models, and other external representations, during instruction and problem-solving activities. Reflecting this common practice, the IES Practice Guide for Developing Effective Fraction Instruction for Kindergarten through 8th Grade (Siegler, Carpenter, Fennell, Geary, Lewis, Okamoto, & Wray, 2010) directly recommends that instructors use visual models to engage students in sense-making activities and to ground their understanding of fraction concepts and procedures. According to the Common Core Standards Writing Team (2013), fractions should be first introduced with visual models in first and second grade under the Geometry strand (e.g., partitioning circles and rectangles into two, three, and four equal shares; 1.G.A.3; 2.G.A.3). The number line is introduced in third grade, when students use partitioning to place fractions on the line. Fourth and fifth graders should learn about multiplication (4.NF.B.4, 5.NF.B.4, 5.NF.B.5, 5.NF.B.6) and division (5.NF.B.3, 5.NF.B.7) with fractions with reference to visual models. Many research-based and empirically-tested effective fraction interventions, such as the Rational Number Project Curriculum (Cramer, Behr, Post, & Lesh, 1997; Cramer, Post, & del Mas, 2002) and several others (e.g., Fazio, Kennedy, & Siegler, 2016; Fuchs et al., 2013; Kellman et al., 2008; Moss & Case, 1999; Rau, Aleven, Rummel, & Pardos, 2014), include visual models as key components. As this body of research and practice recommendations

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demonstrates, there is a great deal of variability in how and which visual models are used, with some interventions (e.g., Rau et al., 2014) including multiple types of visual models, raising the question of whether some types of visual models support children's fraction reasoning better than others.

1.1. Visual models for fractions

In this study, we examine two types of visual models for representing fraction concepts: area models and linear models. Area models are visual models that represent fractions as parts of whole shapes such as circles or rectangles. Typically, whole shapes are partitioned into equal parts, and fraction denominators are represented by the total number of equal parts in each whole shape and fraction numerators are represented by the total number of shaded parts. Area models are quite common; recent empirical studies conducted on children's and adults' reasoning with complex elementary fraction multiplication and division tasks demonstrate a preponderance of area models during instruction (e.g., Baek et al., 2017; Speiser & Walter, 2015; Webel & DeLeeuw, 2016). These types of models are thought to emphasize students' *part-whole* conceptions of fractions (e.g., Kieren, 1976; Wu, 2011), and can successfully support children's visual representation of fractions and understanding the role of common denominators in fraction addition and subtraction (Cramer, Wyberg, & Leavitt, 2008).

Despite their common use, there are many possible limitations of using area models to reason about fraction concepts (e.g., Common Core Standards Writing Team, 2013; Kieren, 1976; Moss & Case, 1999; National Research Council, 2005; Parker & Baldridge, 2004; Wu, 2011). First, given the discrete nature of area models, especially circular area models in which whole units are represented discontinuously by individual shapes, they may prevent children's conceptualization of fractions as a *measurement*. As we discuss in a later section, this may be particularly detrimental for using area models to visually represent certain fraction arithmetic concepts. Second, when representing complex relationships on an area model from a measurement standpoint, fractions and, say, their products refer to different units (i.e., lengths and areas), and, consequently, it is difficult to ascribe meaning to expressions such as $\frac{1}{2} \times \frac{1}{4} + \frac{2}{3}$. Third, given their discrete nature, area models may pose challenges for representing fractions greater than 1, as these representations would necessarily span multiple, and in the case of circular area models, disconnected, shapes. Fourth, area models may be less effective because their part-whole nature may disrupt children's ability to represent two operands on the same visual diagram (e.g., representing 2/3 on circles that have already been partitioned into sixths). Finally, they may be less likely to afford representing each operand in relation to 0 as a common anchor point, preventing children from directly comparing relative magnitudes of operands.

In contrast to area models, linear models, such as the number line, are thought to highlight a *measurement* model of fractions (Kieren, 1976; Moss & Case, 1999) and readily allow children to reason about the magnitude of fractions relative to other rational numbers (Siegler et al., 2011). In particular, fraction multiplication from a measurement perspective provides a more appropriate definition of multiplication (as scaling) that applies to all rational numbers, overcoming conceptual limitations associated with area models (e.g., repeated addition). Furthermore, number line models support awareness of units, which is central to the development of deep understanding of fraction operations. Students learn that adding and subtracting fractions involving like and unlike denominators require the same initial process of constructing common units, and that multiplying and dividing fractions generate compound units. Siegler et al. (2011) have argued that number line models are a critically important tool for reasoning about fractions, and more generally, all rational numbers. Indeed, in studies of children's whole number magnitude reasoning, children are better able to reason about whole number arithmetic when addend magnitudes were represented on number lines (e.g., Booth & Siegler, 2008). Similarly, learning about fractions using number lines can result in better understanding of the relative magnitude of fractions (e.g., Fazio et al., 2016) and children who can place fractions on number lines with better precision are also more likely to have better fraction arithmetic skills (Siegler & Pyke, 2013; Siegler et al., 2011). Given this empirical evidence suggesting that number lines can be an effective external visual representation for reasoning about and understanding fraction magnitudes, and relative magnitudes of fractions, the IES Practice Guide specifically recommends using number line models as a central representational tool.

1.2. Comparing visual models

Despite these theoretical arguments, research, and recommendations pointing towards the utility of number lines as an effective visual model for understanding fractions, and the many potential pitfalls of area models, recent studies with elementary students remain inconclusive about the most optimal visual model for learning fraction concepts (Cramer & Wyburg, 2009; Wilkerson et al., 2015). Few studies have directly compared the relative benefits and limitations of using these two types of visual models during learning or problem solving. Here, we argue that area models and linear models, specifically number line models, likely do have differential effects on children's fraction reasoning due to different affordances of these representations.

Children's behavior is inherently variable (Siegler, 1996). Problems with different features often afford different strategies; for example, empirical research clearly demonstrates that specific features of fraction tasks contribute to intraindividual variability in students' approaches (e.g., Alibali & Sidney, 2015; Fazio, DeWolf, & Siegler, 2016; Schneider & Siegler, 2010). Visual models can highlight different aspects of complex conceptual relationships (see Rau & Matthews, 2017 for a discussion, Ainsworth, 2006), and even minor perceptual differences in visual models for fractions may elicit slightly different ways of reasoning, and approaches to problem solving, that are more or less accurate. For example, in one study of young children's proportional reasoning, Boyer, Levine, and Huttenlocher (2008) compared children's ability to match proportions across visual representations in which proportions were represented with two differently-colored continuous areas or partitioned areas. When visual representations of mixtures included discrete partitions, these visuals elicited counting-based strategies, disrupting children's ability to match based on overall proportion. Thus, this relatively minor perceptual difference, including partitions or not, shaped children's strategies for reasoning in this task.

Two recent experimental studies (Hamdan & Gunderson, 2017; Kaminski, 2018) have compared children's learning about fraction magnitudes and fraction addition using circular area models, linear models (e.g., number lines), or no model at all. In line with what several researchers have proposed (e.g., Moss & Case, 1999; Siegler et al., 2011), Hamdan and Gunderson observed a number line advantage. Students who were trained to use a number line to represent fraction magnitudes were more successful on a later, symbol-only fraction comparison task than students who were trained to use circular area models. In contrast, across two experiments, Kaminski (2018) found no advantage of visual models for understanding fraction addition in comparison to instruction without visual models, and even observed some detrimental effects of using number lines for learning fraction addition.

Across this limited evidence, it remains unclear whether number lines better support children's understanding of fractions, and fraction arithmetic, and under what conditions. Furthermore, as demonstrated by Kaminski (2018) findings, although external visual models can help learners to generate internal, mental models of complex relationships (e.g., Butcher, 2006) and support accurate problem solving (e.g., Cooper, Sidney, & Alibali, 2018; Larkin & Simon, 1987) in comparison to text alone with no visual model, including diagrams in practice and instruction does not always lead to increased learning (e.g., Kaminski & Sloutsky, 2013; Bergey, Cromley, Kirchgessner, & Newcombe, 2015). Finally, because the studies conducted by Hamdan and Gunderson (2017) and Kaminski (2018) have focused primarily on fraction magnitudes and simple, common denominator fraction addition, little is known about how visual representations support, or fail to support, children's effective reasoning about other arithmetic operations.

1.3. Fraction division

In the current study, we focus on fraction division given children's (Sidney & Alibali, 2015, 2017) and even adults' (Ma, 1999; Sidney, Hattikudur, & Alibali, 2015) relatively poor performance on common measures of conceptual understanding of fraction division. Within rational number arithmetic, division is arguably the most difficult to represent and the least well understood arithmetic operation (see Ball, 1990; Dixon, Deets, & Bangert, 2001; Ma, 1999; Sidney, Chan, & Alibali, 2013; Siegler et al., 2011). Thus, we targeted fraction division with an aim to elicit sound conceptual models of the most challenging fraction concept covered in elementary school mathematics.

Two common conceptual models for reasoning about division are *quotitive models* of division and *partitive models* of division. When representing a quotitive model of division, children must reason about the magnitude of the first operand (i.e., the dividend), the magnitude of the second operand (i.e., the divisor), and the quotient as an indicator of the relative size of the divisor to the dividend (i.e., how many times does the divisor "fit" into the dividend). In a partitive model of division, children also need to represent the magnitude of the first operand (i.e., the dividend), but represent the magnitude of the second operand (i.e., the divisor) as a number of equal groups or segments and the quotient as the magnitude of each group or segment. Although children use both partitive and quotitive models to reason about whole number division (Sidney et al., 2013), children tend to rely on quotitive models of division when reasoning about fraction division (Fischbein, Deri, Nello, & Marino, 1985).

Given the nature of quotitive models of division, we expected that number lines would better elicit sound conceptual models of fraction division than area models. First, quotitive models of division, also called *measurement* models of division (e.g., Cramer, Monson, Whitney, Leavitt, & Wyberg, 2010), rely on the conceptualization of fractions as a measurement, which number lines are thought to highlight (e.g., Moss & Case, 1999). Second, not only do linear models, such as number lines, appear to effectively support children's reasoning about individual fraction magnitudes (e.g., Moss & Case, 1999; Siegler et al., 2011; Fazio et al., 2016), which is a fundamental first step towards constructing a conceptually-sound visual model of division, we also expected that they would also afford reasoning about the *relative* magnitudes of operands. A number line allows students to represent more than one numerical magnitude, or operand, on a single, common scale (e.g., both 4 and 1/5 from the fraction division problem $4 \div 1/5$ can be represented relative to 0 on a single number line that ranges from 0 on the left to 6 on the right), with 0 as a common anchor point. This may better allow learners to directly perceive and compare the relative magnitudes of the dividend and the divisor, which may afford more accurate reasoning about the quotient as well.

1.4. Current study

The primary goal of the current study was to examine whether the nature of visual models shape children's ability to reason about the conceptual relationships between dividend, divisor, and quotient in fraction division. Furthermore, given the mixed evidence for the utility of visual models for learning about fraction arithmetic (Kaminski, 2018), we examined whether visual models do provide support for children's emerging fraction division understanding in contrast to reasoning without visual models. Thus, in the current study, we investigated whether asking children to solve fraction division problems using a number line, an area model, or no visual model at all resulted in more conceptually-sound, and successful approaches to reasoning about fraction division. To capture their conceptual understanding and interpretation of the relationship between dividend, divisor, and quotient, we inspected whether children's written work on researcher-provided visual models reflected conceptually-sound models of division. To capture children's problem-solving success, we measured children's generation of a correct quotient, regardless of the child's solution method. Critically, we observed children's approaches to fraction division understanding near the beginning of their business-as-usual fraction division instruction, before learning about complex fraction division (i.e., division with a proper fraction or mixed number). This allowed us to observe the differential effects of visual models on children's emerging ideas about fraction division, rather than rote knowledge of procedures, such as invert-and-multiply.

1.5. Hypotheses

We hypothesized that asking children to use visual models to reason about fraction division would be beneficial for their problem solving and reasoning. First and foremost, we expected a *number line advantage*, such that children who solved problems with a number line would have higher accuracy rates, and be more likely to generate conceptual models of division, than children in any other experimental condition (*H1a*). Among the remaining children, we expected a visual model advantage such that those who solved problems with an area model would have greater success than those who solved problems with no visual model (*H1b*). Finally, we planned to compare children's accuracy in each area model condition, rectangular area and circular area, to examine whether the more linear, continuous rectangular model would support accuracy in comparison to discrete circles (*H1c*).

Furthermore, we hypothesized that the number line advantage was due in part to differences in how number lines and circles afford representing the given operands (*H2*). We expected that some children, particularly those provided with circles and rectangles, might choose to represent each operand on different parts of the diagram (i.e., on different circles), which might disrupt their ability to reason about the relative magnitudes of the two operands. In contrast, we expected that children would be less likely to represent magnitudes on two separate, non-overlapping segments of the number line diagrams, given that they afford representing each magnitude relative to the same, common endpoint (i.e., starting at 0).

Additionally, we explored our data in two ways. First, we explored whether children's confidence and perceptions of difficulty differed across visual models. Although we expected number lines to afford sound conceptual reasoning about fraction division, we expected that children might be more familiar with area models, and thus more confident in their performance, since students in the US are introduced to dividing area models into equal fractional shares as early as first grade (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). Given previous research (Wall, Thompson, Dunlosky, & Merriman, 2016) that showed children were more accurate and confident when estimating the magnitudes of numbers within smaller, more familiar numerical ranges than when estimating numbers within larger, less familiar numerical ranges, we sought to explore the possibility that children may be more confident, and report less difficulty, with more familiar area models. Second, we explored whether the diagrams used during the focal problem-solving task affected children's reasoning on two other types of tasks designed to assess conceptual knowledge of fraction division: story generation and story problem-solving.

2. Method

2.1. Power analysis

To determine our necessary sample size, we ran an *a priori* power analysis based on our initial plan of using ANCOVA to examine the effect of experimental visual model condition on children's performance during the focal task. Based on prior work examining the role of diagrams in learning (e.g., Beitzel & Staley, 2015; Butcher, 2006; Moreno, Ozogul, & Reisslein, 2011), we anticipated that differences in visual model condition would explain about 6% of unique variance in performance, ${\eta_p}^2 = 0.06$, over and above our covariates. Using the *pwr* package (Champely, 2015) in *R*, we determined that a sample size of 120 participants was necessary for 80% power to detect an effect of this size for a 1 *df* test in a model that included fixed effects of condition and three covariates (grade, gender, and problem order).

2.2. Participants

Participants were 123 children in late Spring of 5th grade or Fall of 6th grade (*M* age = 11.6*y*, *SD* = 1.4*y*; 45.6% girls; 75.0% White) from one public intermediate school in the midwestern United States. State standards for mathematics education are aligned with the CCSSM, with fraction division first introduced in 5th grade. A small proportion of children at this school qualify for the free and reduced-price lunch program (18.30%). We obtained mathematics achievement data from the Spring before study enrollment for 113 participants, missing data was primarily from children who had not completed a standardized assessment the prior Spring due to transferring into the district.

2.3. Tasks

2.3.1. Diagram task

Each child was randomly assigned to one of four between-subjects visual model conditions (see Fig. 1) as they solved 18 fraction division problems: (a) *circular* area (n = 33), (b) *rectangular* area (n = 29), (c) *number line* (n = 31), and (d) *no visual model* provided (n = 30). Each provided diagram represented six whole units (e.g., six circles) that were partitioned into the denominator units of the divisor. By providing partitioned diagrams, we facilitated students' use of the diagrams and controlled for differences in students' ability to represent fraction magnitudes precisely. The problems included unit fraction, proper fraction, mixed number, and whole number operands, allowing us to examine whether the effects of diagram varied across problem type (see Table 1). Each problem was presented on a separate page and in one of two predetermined random orders across all conditions; half of children were assigned to each order. After solving each problem, children rated their confidence in solving each problem (*How confident are you that you solved the problem correctly*? 0% *definitely did not* − 100% *definitely did*) and the difficulty level of each question (*How difficult was it to answer this problem? not difficult at all* [1] to *very difficult* [4]) on the bottom of that problem's page.

2.3.2. Far transfer tasks

Participants generated stories to represent two fraction division problems (e.g., Write a story problem that represents 4 ÷ 1/5.) (see Sidney & Alibali, 2015; Sidney et al., 2015) and also solved two fraction division story problems (e.g., Shay wants to spend half of her summer volunteering for charities. If she wants to spend an equal amount of time volunteering for three charities, what fraction of her summer will she spend volunteering for each charity?).

2.4. Procedure

Participants were tested individually with a trained postdoctoral (first author) or undergraduate researcher in a quiet location in their school. The Diagram Task was introduced with a script printed on the first page of each packet and read aloud by the researcher. We began by telling children that we were interested in the strategies they used to solve "*new kinds of math problems*", and asking children to show their work for each problem. Then, the researcher demonstrated "*how you can show your work*" using a whole number division example (6 ÷ 2). In the example, the researcher spoke about making "*a group of six*" and showing or thinking about "*how big six is*", making "*a group of two*" and showing or thinking about "*how big two is*", and finding "*how many times a group of two goes into a group of six*". In the diagram conditions, the researcher drew a diagram using the same type of diagrams included in the child's packet, demonstrating how to represent the dividend,

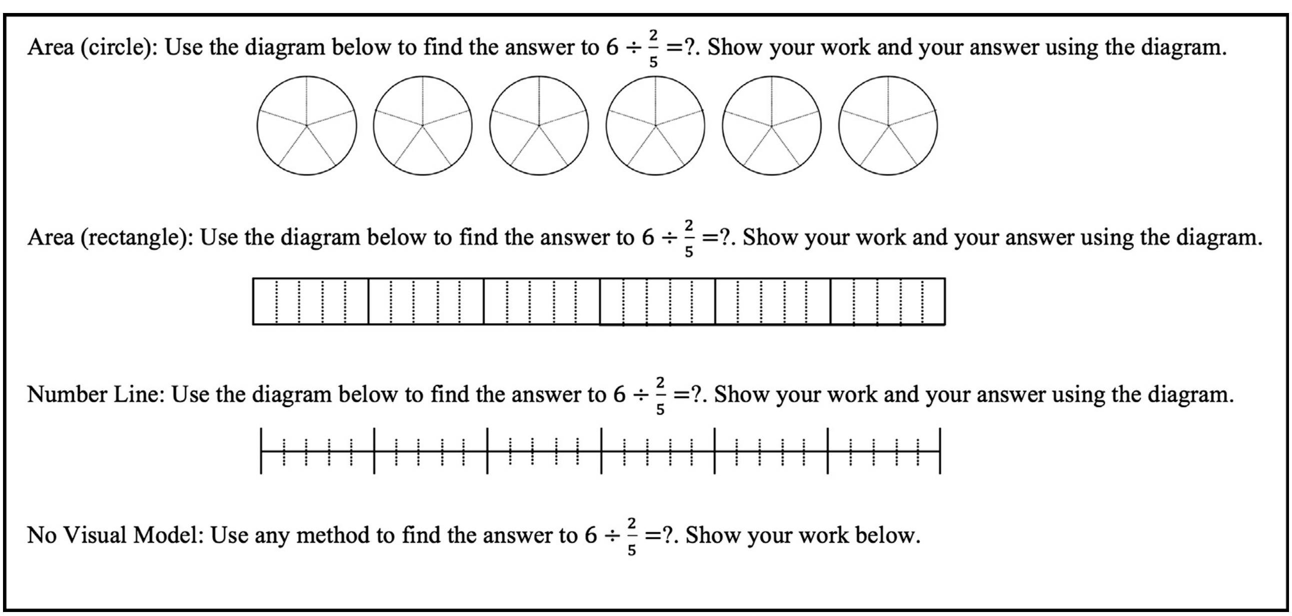

Fig. 1. Example problems from each condition. One example fraction division problem illustrated in each of the four between-subjects experimental conditions: circular area model, rectangular area model, number line model, and no visual model (from top to bottom). Each child solved 18 problems in their visual model condition.

Table 1
Examples of Fraction Division Problems by Problem Type and Accuracy by Condition.

				% of Problems Correctly Answered				
Dividend	Divisor	Problem 1	Problem 2	Avg.	Circ.	Rect.	NL	None
Unit fraction	Unit fraction	1/3 ÷ 1/9 = ?	1/4 ÷ 1/8 = ?	**41%**	36%	30%	56%	38%
Proper fraction	Unit fraction	4/5 ÷ 1/5 = ?	2/3 ÷ 1/6 = ?	**42%**	40%	38%	55%	35%
Whole number	Unit fraction	6 ÷ 1/3 = ?	4 ÷ 1/4 = ?	**39%**	40%	29%	52%	33%
Mixed number	Unit fraction	2 1/4 ÷ 1/4 = ?	3 3/5 ÷ 1/5 = ?	**33%**	32%	32%	46%	23%
Proper fraction	Proper fraction	2/3 ÷ 2/9 = ?	3/4 ÷ 3/8 = ?	**39%**	35%	26%	54%	37%
Whole number	Proper fraction	6 ÷ 2/3 = ?	2 ÷ 2/5 = ?	**41%**	33%	42%	61%	25%
Mixed number	Proper fraction	5 1/3 ÷ 2/3 = ?	4 1/6 ÷ 5/6 = ?	**31%**	26%	36%	45%	18%
Unit fraction	Whole number	1/3 ÷ 2 = ?	1/2 ÷ 4 = ?	**23%**	16%	14%	28%	33%
Proper fraction	Whole number	2/5 ÷ 4 = ?	3/4 ÷ 6 = ?	**15%**	16%	6%	10%	27%
		Overall Percent Correct		**33%**	31%	28%	45%	30%
		Average Confidence Rating		**68%**	77%	58%	72%	62%
		Average Difficulty Rating		**1.94**	1.64	2.16	1.98	2.03

divisor, and quotient on the given diagram. In the *no diagram* condition, the experimenter simply wrote the numerals '6' and '2'. Apart from instructions to "*show*" (diagram conditions) or "*think about*" (no diagram condition) the magnitude of the numbers, the verbal description of the example and the number of times the researcher paused to draw or write were identical in all conditions.

We chose to model a whole number division problem for children given previous research (Sidney & Alibali, 2017) suggesting that children are more likely to successfully model fraction division immediately after modeling whole number division. We chose to model quotitive division given that children reason with both quotitive and partitive models when demonstrating whole number division (e.g. Sidney & Alibali, 2013), but they appear to favor quotitive division when reasoning about division by a fraction (e.g., Fischbein et al., 1985; Sidney & Alibali, 2017) and find quotitive division problems easier to interpret and model than partitive division problems (English & Halford, 1995; Watanabe, Lo, & Son, 2017; Zambat, 2015). Participants completed all problems without feedback from the experimenter.

2.5. Coding children's work

We coded for children's accuracy and conceptual models on each problem. Accuracy was defined as whether or not the child wrote the correct answer to each problem somewhere on its page. All completed problems were double-coded by two independent coders with high agreement (agreement on 97.0% of trials). Disagreements often occurred when children's handwriting was poor, or when they generated both a correct and incorrect answer on their page. All disagreements were flagged and resolved through discussion until 100% agreement was reached.

Furthermore, to examine the nature of the visual models that children generated in their drawings, two coders coded children's final written work on each problem. Our coding scheme was adapted from one used in prior work (Sidney & Alibali, 2017; Sidney, 2016) and was aimed at categorizing children's overt strategies on fraction division problems. First, we coded whether children accurately represented the magnitude of each operand. Then, we coded the nature of the relationship between the operands (see Table 2). For example, some children represented a division relationship, while others represented other operations such as subtraction or addition, and still others drew incomplete diagrams, for example, diagrams that only included representations of each operand but did not represent the relationship between operands. Critically, children's work on each problem was categorized as reflecting quotitive division, partitive division, or neither (see Fig. 2 for examples of student work). "Quotitive division" was coded when children fully partitioned the dividend into segments as large as the divisor (see Fig. 2, Panel A). "Partitive division" was coded when children divided the dividend equally into a number of segments specified by the divisor (see Fig. 2, Panel B). All other models were classified as "other" (see Fig. 2, Panels C–F). Agreement for division model coding was also high (94.4% agreement). All disagreements were flagged and resolved through discussion until 100% agreement was reached.

Finally, to examine whether number line models and circular area models afford different types of spatial relationships when representing two operands on the same diagram, we coded whether or not the operands were represented on separate, non-overlapping parts of the diagram (see Fig. 2, Panel E for an example). For example, many children used different circles to represent the magnitudes of the dividend and the divisor. Some children even drew additional circles to do so. Two independent coders double-coded half of the data, and agreement was high (99.7%). To analyze this data, we distinguished between children who never represented operands separately and children who did so on at least one trial.

3. Results

3.1. Random assignment

Children's achievement scores did not differ across conditions, $F(3, 109) = 1.16$, $p = .33$, and the distribution of boys and girls did not differ across conditions, $\chi^2(3, N = 123) = 0.60$, indicating successful random assignment to experimental condition.

3.2. Analytic overview

When conducting our power analysis, we had planned to use ANCOVA to examine our data. However, we revised our planned analysis to account for the non-independence due to item (see Barr, Levy, Scheepers, & Tily, 2013) and to account for variability in problem type. We used two logistic mixed effect regression models to test our primary hypotheses about children's accuracy and conceptual models on the diagram task using the *lme4* package (Bates, Maechler, Bolker, & Walker, 2015) in *R*. Each model estimated fixed effects of visual model condition, problem type, problem order, grade, and gender, the by-participant and by-item random intercepts, the by-participant random slope of divisor type, and the by-item random slope of condition.[1] By including both by-participant and by-item random effects, these models

[1] The de-identified coded dataset, coding manuals, and *R* script for the analyses are publicly available on Open Science Framework: https://osf.io/vq6cz/.

Table 2
Coding Conceptual Models from Diagrams.

Model	Model Type	Description (*Op1*, *Op2* = ?)	Examples (4 1/6 ÷ 5/6 = ? & 3/4 ÷ 6 = ?)
Quotitive Division	Division: Correct	The magnitude of operand 1 (*Op1*) is represented and divided into groups as big as operand 2 (*Op2*)	Drawing shows 4 1/6 divided into 5 groups of 5/6 (see Fig. 2A)
Partitive Division	Division: Correct	The magnitude of *Op1* is represented and divided into the number of groups specified by *Op2*	Drawing shows 3/4 divided into 6 equal parts (see Fig. 2B)
Switched Numbers	Division: Incorrect	A quotitive or partitive division model representing *Op2*, *Op1*	Drawing shows 6 divided into 8 groups of 3/4
Wrong Numbers	Division: Incorrect	A quotitive or partitive division model with incorrect operands	Drawing shows 40 divided into 8 groups of 5
Multiplication	Other Operation	*Op2* groups of *Op1*; *Op1* groups of *Op2*	Drawing shows 3/4 iterated 6 times
Subtraction	Other Operation	One group of *Op2* is represented within *Op1* & either the remainder of *Op1* is highlighted as the answer or written work includes a subtraction symbol	Drawing shows 4 1/6, one group of 5/6 is represented within 4 1/6, the remaining area is bracketed (or labeled, see Fig. 2C)
Addition	Other Operation	*Op1* and *Op2* are represented as adjoining magnitudes and written work includes an addition symbol	Drawing shows one segment of 4 1/6 and one adjoining group of 5/6. Above the diagram, the written work shows “4 1/6 + 5/6 = 5”
Discrete Units	Other Operation	One (or more) groups of *Op2* are highlighted in each whole unit, without fully accounting for the magnitude of *Op1*	Drawing shows 4 1/6; one group of 5/6 is drawn in each whole unit, for a total of 4 groups of 5/6 (see Fig. 2D)
Numbers Only	Numbers Only	The magnitudes of *Op1* and *Op2* are represented, but the written work does not represent a mathematical relationship between their magnitudes	Drawing shows one group of 4 1/6 and one group of 5/6 (see Fig. 2E)
Answer Only	No Model	Only the magnitude of the quotient is represented on the diagram	Drawing shows the magnitude of 5
No Model	No Model	Only one operand is represented or the diagram is blank	Drawing shows the magnitude of 5/6

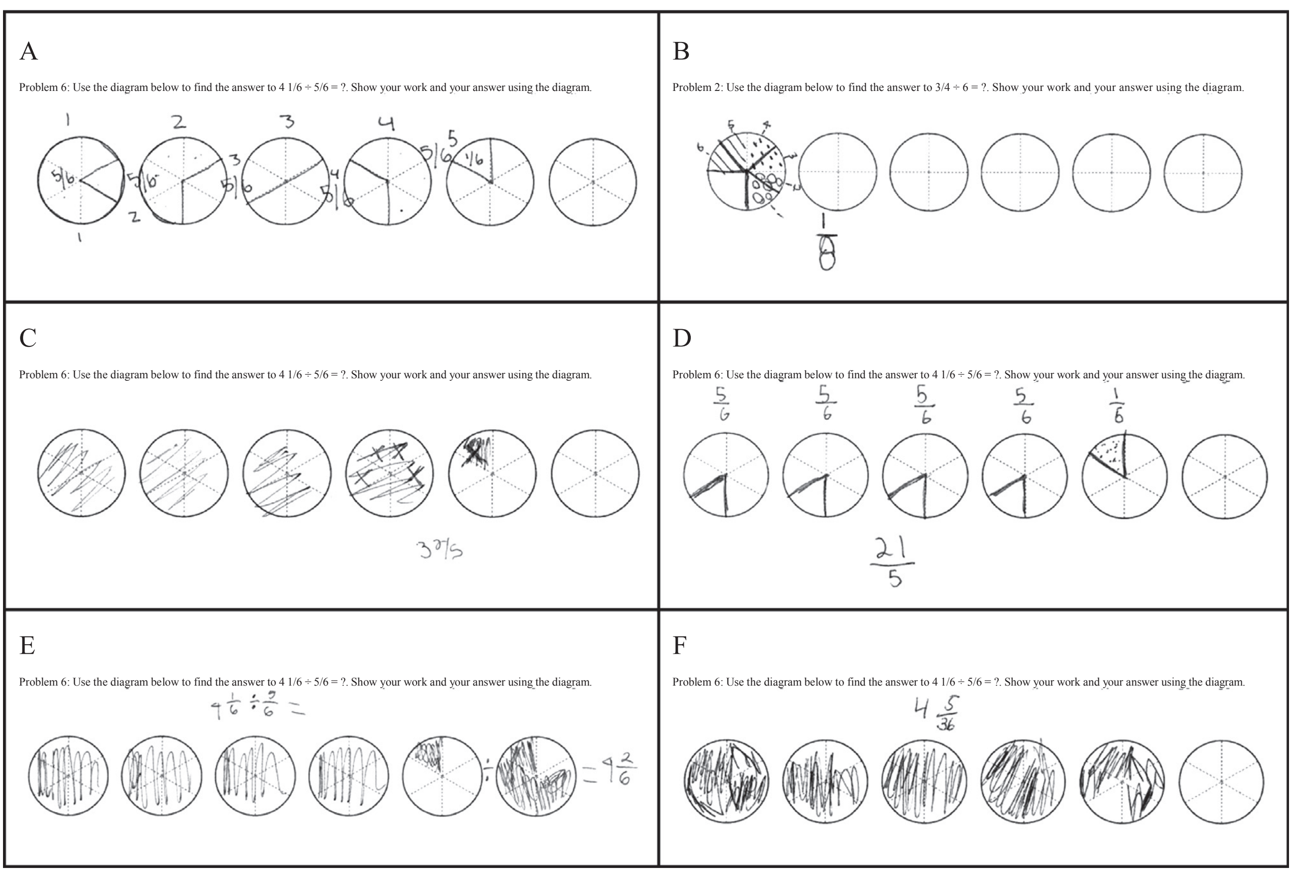

Fig. 2. Examples of student work. Panels A and B reflect accurate *quotitive* and *partitive* conceptual models of division, respectively. Panel C shows an inaccurate *subtraction model*, Panel D shows an inaccurate *discrete units model*, Panel E shows an inaccurate *numbers only model*, and Panel F shows a drawing coded as *no model.*

allow us to account for non-independence due to items (across participants) and participants (across items). Although Barr and colleagues (2013) recommend estimating all possible random slopes, maximal models failed to converge; therefore, we did not estimate random slopes for the covariates.

To represent our detailed hypotheses for visual model condition, this factor was represented by a set of three contrast variables, each representing a specific hypothesis: number line condition vs. all other conditions (*H1a*), area model conditions vs. no visual model (*H1b*), and circular area model vs. rectangular area model (*H1c*). We defined problem type as the type of divisor in the problem (i.e., unit fraction, proper fraction, and whole number), and this factor was represented by a pair of contrast variables: whole number vs. fraction divisor, and unit fraction vs. proper fraction divisor. We report Type III Wald χ^2 for each parameter, and the change in Akaike's Information Criteria (ΔAIC) for each model, as compared to the null model for each outcome.

3.3. Accuracy

As hypothesized (*H1a*), children in the number line condition were more likely to accurately solve each fraction division problem than children in any other experimental condition, $OR = 3.28$, $\chi^2(1) = 5.06$, $p = .02$. Children who solved problems in the number line condition generated correct answers on about 45% of problems, on average, whereas children who solved problems in the other conditions generated correct answers on around 30% of problems (see Table 1). Somewhat unexpectedly (*H1b*), children who solved problems with circular area models or rectangular area models were no more likely to be accurate on each problem than children who solved problems without visual models, $OR = 1.61$, $\chi^2(1) = 0.71$, $p = .40$. Finally (*H1c*), children who solved problems with rectangular area models were no more likely to be accurate than children who solved problems with circular area models. $OR = 0.64$, $\chi^2(1) = 0.53$, $p = .47$.

Furthermore, children's likelihood of accuracy differed across problem type, such that children were more accurate on problems with fraction divisors than whole number divisors, $OR = 17.59$, $\chi^2(1) = 10.80$, $p < .01$. Children were equally accurate on problems with unit fraction divisors and proper fraction divisors, $OR = 1.48$, $\chi^2(1) = 1.68$, $p = .19$. On average, children only solved 19% of whole number divisor problems correctly, as compared to about 44% of fraction divisor problems (see Table 1). There was no significant effect of problem order, $\chi^2(1) = 1.46$, $p = .23$, or gender, $\chi^2(1) = 1.00$, $p = .32$. Overall, sixth graders were more likely to be accurate than fifth graders, $OR = 4.64$, $\chi^2(1) = 12.31$, $p < .01$. Model AIC was reduced relative to the null model, $\Delta AIC = -195.00$.

3.4. Conceptual models

We analyzed the likelihood of generating a sound conceptual model of division, either quotitive or partitive, among children in the visual model conditions. Children in the no visual model condition were not provided with a visual model nor instructed to show their work on a diagram. There were almost no instances of spontaneously drawing conceptually sound models of division in the no visual model condition (3% of all trials), thus these children were excluded from the conceptual model analysis altogether.

In line with our focal hypothesis (*H1a*), and observed patterns of accuracy, children who were provided with number line models when reasoning about fraction division were considerably more likely to generate a conceptually-sound model of division on any given problem than children in either area model conditions, $OR = 4.64$, $\chi^2(1) = 7.71$, $p < .01$. Children in the number line condition generated division models on 52% of trials, whereas children in the rectangular area condition generated division models on 35% of trials and those who were given circular area models generated division models on only 22% of trials. The likelihood of generating a conceptual model of division was no different in each area model condition, $OR = 1.88$, $\chi^2(1) = 0.46$, $p = .50$.

In line with accuracy rates, children were more likely to generate conceptual models on fraction divisor problems than whole number divisor problems, $OR = 29.39$, $\chi^2(1) = 7.64$, $p < .01$, but equally likely on unit and proper fraction problems, $OR = 1.04$, $\chi^2(1) = 0.01$, $p = .91$. Problem order, $\chi^2(1) = 0.93$, $p = .34$, and child gender, $\chi^2(1) = 1.15$, $p = .22$, did not predict generating conceptual models; however, sixth graders were more likely to generate conceptual models of division, $\chi^2(1) = 6.18$, $p = .01$. Model AIC was reduced relative to the null model, $\Delta AIC = -222.00$.

To provide some insight on the number of children who may have experienced a number line advantage, we also examined the percentage of children in each condition who *consistently* generated sound conceptual models on a majority of trials. The majority (71%) of children who demonstrated their work on a number line consistently drew accurate models of either quotitive or partitive division. In contrast, many fewer did so in the circular (21%) or rectangular (34%) conditions. A logistic regression analysis revealed the odds of consistently generating a division model among children in the number line condition were nearly 10 times greater, $OR = 9.83$, $\chi^2(1) = 16.01$, $p < .01$, than among children in the circular condition, and over five times greater, $OR = 5.30$, $\chi^2(1) = 8.60$, $p < .01$, than those in the rectangular condition, controlling for problem order, $p = .96$, gender, $p = .04$, and grade, $p = .02$. Model AIC was reduced relative to the null model, $\Delta AIC = -16.18$.

3.5. Representing magnitudes

As expected, number lines supported children's generation of conceptually-sound models of fraction division. Thus, we further tested our hypothesis that the number line advantage may be due in part to differences in how children represent the relative magnitudes of each operand on the provided diagram. We examined the number of children in each visual model condition who ever represented the operands separately, rather than overlapping on the same whole unit, and found a significant effect of condition, $\chi^2(2, N = 93) = 23.16$, $p < .01$. No child in the number line condition drew operands on separate segments of the number line; when children represented both operands, they always represented them relative to the same, common endpoint. In contrast, about one-half (15/33) of the children in the circular area condition represented the operands on separate circles on at least one trial, and about one-third (9/29) of the children in the rectangular area condition represented the operands on separate rectangles at least once.

Given these patterns, we also explored whether children's rates of representing operands separately were related to their rates of generating conceptually sound models of division, among children in the area model conditions using a general linear model. Indeed, children who represented operands separately on a greater proportion of trials generated fewer conceptual models of fraction division, $b = -0.02$, $t(56) = -3.04$, $p < .01$, controlling for area model condition, $p = .58$, order, $p = .65$, gender, $p = .20$, and grade, $p = .02$; total $R^2 = 0.31$.

3.6. Confidence and difficulty ratings

Next, we examined children's confidence ratings, which ranged from 0 (*definitely did not [solve correctly]*) to 100 (*definitely did*), $M = 65.49$, $SD = 23.15$, and difficulty ratings, which ranged from 1 (*not difficult at all*) to 4 (*very difficult*), $M = 1.94$, $SD = 0.80$. Confidence and difficulty ratings were analyzed using linear mixed effects models similar in structure to those used for accuracy and conceptual models, but with visual model condition represented by a different set of contrast variables. Because we observed that children in the circular area condition had, on average, slightly higher ratings of confidence and lower ratings of difficulty than the other children (see Table 1), our contrast codes represent pairwise comparisons between the circular area condition and each remaining condition. We report the Type III Wald F tests using the Kenward-Roger approximation as implemented by *car* (Fox & Weisberg, 2011).

In contrast to patterns of accuracy, there was no significant effect of visual model condition on children's confidence ratings, $F(3, 120.17) = 2.49$, $p = .06$, and no effects of problem order, $p = .62$ or gender, $p = .80$. In line with patterns of accuracy, there was a significant effect of divisor type, $F(2, 32.67) = 5.94$, $p < .01$, such that children were more confident on problems with a fraction divisor than those with a whole number divisor, $F(1, 47.52) = 11.75$, $p < .01$. Also, children in 6th grade were more confident than those in 5th grade, $F(1, 123.97) = 4.08$, $p < .05$. There were no differences in children's ratings of difficulty across visual model condition, $F(3, 20.32) = 2.88$, $p = .06$, and no effects of divisor type, $p = .24$, problem order, $p = .73$, grade, $p = .07$, or gender, $p = .71$.

3.7. Far transfer tasks

Finally, we explored children's performance on four far transfer tasks, two story generation tasks and two story problem solving tasks, to examine whether visual model condition during earlier problem-solving had any lasting effects on children's performance on other tasks designed to assess conceptual understanding of fraction division. To analyze these data, we fit separate logistic regression models regressing accuracy on visual model condition, grade, and gender. Children's performance on each item did not vary by condition, $0.55 < \chi^2(3) < 6.56$, $0.09 < ps < 0.93$, $94 < Ns < 104$, suggesting that the effects of visual models on children's conceptual understanding of fraction division were constrained to problems with which those visual models were presented. In other words, children who solved problems with number lines, who were likely to display conceptual understanding of fraction division in their drawings using number lines, were no more likely to generate conceptually-sound solutions to far transfer items when diagrams were no longer present. Furthermore, we noted that although several students drew diagrams to help support their story problem-solving ($n = 30$), only seven children drew number line diagrams.

4. Discussion

4.1. Summary of findings

As expected, children who were asked to demonstrate their reasoning on number lines when solving fraction division problems were more accurate problem-to-problem than were children asked to use area models or no visual model at all. However, somewhat unexpectedly, we did not observe any advantages of using area models to show work during fraction division problem solving. Instead, problem-to-problem accuracy was similar across the two area model conditions (i.e., circular and rectangular), and no higher than for those children who solved fraction division problems without a visual model. Further, this pattern of results, revealing a number line advantage, was also apparent for children's likelihood of producing a sound conceptual model of fraction division. Strikingly, over twice as many children who solved problems with number lines consistently drew sound conceptual models of division across different types of problems than in the other visual model conditions. Sixth graders were more accurate, confident, and likely to produce sound conceptual models of fraction division than were fifth graders.

4.2. Visual models for fractions

Overall, our findings suggest that visual models, such as diagrams, can support children's thinking about the conceptual structure of division, however, not all visual models are equally effective at supporting conceptual understanding. Number lines, as compared to area models or no visual models at all, elicited conceptually accurate strategies across a range of problem types and consistently across the majority of problems. Our qualitative analysis of children's diagrams suggests that one reason that the number line may have afforded an advantage over the area models is that children were more likely to accurately represent the magnitude of each operand relative to the same common endpoint, as opposed to representing the operands on separate circles or rectangles.

One potential limitation of this finding is that the visual models we provided were pre-segmented into divisor units to take some of the drawing demands off of the children so that they could more effectively show their fraction division understanding. This pre-segmentation may have led participants to rate the circle problems as less difficult than they would have rated unsegmented circles given that research in mathematics education (Myers, Confrey, Nguyen, & Mojica, 2009) has shown that children find it particularly difficult to partition circles. Future research could investigate the impact of asking students to construct units on empty or unsegmented number lines or unpartitioned whole shapes on fraction division problem solving.

All the children in our sample viewed a whole number division example, using a visual model consistent with their condition assignment, prior to completing the fraction division diagram problem-solving task. Prior research has demonstrated that children in 5th and 6th grade are often more successful at modeling fraction division immediately after modeling whole number division (Sidney & Alibali, 2015, 2017). Because we reminded children of quotitive division with whole numbers, our findings may provide evidence that children who reasoned with number line diagrams were more likely to *transfer* appropriate ways of reasoning about division from the whole number example to the fraction division problems.

Although children often have an underdeveloped conceptual understanding of fraction division (e.g., Mack, 2001; Sidney & Alibali, 2017), many children in our sample, and particularly those who reasoned with a number line, generated conceptual models of fraction division that mirrored our whole number example: they represented the magnitude of the first operand, then represented the magnitude of the second operand, and finally drew as many groups or segments as big as the second operand that "fit" in the first operand. This evidence of transfer may be one promising sign of integration across whole numbers and fraction concepts (see Siegler et al., 2011). Future work could more closely examine whether the type of example problem (quotitive vs. partitive; whole number vs. fraction division vs. other fraction operation; whole number vs. fraction divisor) differentially impacts problem-solving performance, and whether analogical transfer underlies the number line advantage in the current study.

4.3. Limitations

Here, we have demonstrated that there is a number line advantage for fraction division problem solving. However, the current study leaves open the question of whether number lines also uniquely afford benefits when students *learn* about fraction division in the lab and in real classrooms. Although this study does suggest that number lines may support children's emerging fraction division conceptualizations, it remains an open question as to which order the visual models should be introduced (e.g., number lines first then area models next) during classroom instruction to optimize student learning. For example, our study does not rule out the possibility that area models may have learning benefits, perhaps when included along with number lines.

Furthermore, we focused on fraction division, given its difficulty, limiting the findings' generalizability to children's learning of other fraction operations. For example, our findings stand in contrast to Kaminski (2018) study of 3rd grade children's use of visual models when learning about fraction addition, in which there was no number line advantage and some evidence of a number line *disadvantage*. One possible reason for these contrasting findings may be related to differences in the ages of the participants. Area models may be more familiar than number lines among third graders given that they are introduced earlier in Common Core-aligned curricula (as early as 1st grade under the Geometry strand 1.G.A.3), though age alone seems unlikely to explain this difference given the number line advantage among similarly aged children learning fraction magnitudes (Hamdan & Gunderson, 2017). Another possibility is that differences may arise from differences in how the number line was used in these studies. Number lines may be most advantageous when used to highlight numerical magnitude as a distance from zero, as we have done in our example problem in the current study.

A third possibility is that visual models afford different approaches to reasoning and representing the conceptual structure of arithmetic operations, and addition and division benefit from different affordances. In quotitive division, the ability to compare the relative magnitudes of each operand is a key component to representing the division relationship, and our qualitative analysis suggests that number lines afford representing the operands in ways that allow this comparison process to take place. Comparing the relative size of two operands may be helpful for subtraction as well, but may not be as helpful for addition and multiplication. For example, if a child conceptualizes fraction multiplication as a *scaling* operation (e.g., $5 \times 1/4$ as finding a magnitude that is ¼ the size of 5), area models may better support this conceptualization. Thus, we view this work as only a first step towards understanding the affordances of various visual models for supporting key ways of conceptualizing fraction arithmetic. Because the conceptual models themselves are varied across operations, we expect that the effects of visual models may also vary across arithmetic operations. If this is the case, it would also raise new questions about the optimal times to introduce each visual model to children across the entire span of fraction instruction.

Finally, based on prior work linking confidence and familiarity in math (Fitzsimmons, Thompson, & Sidney, in preparation; Wall et al., 2016), we sought to explore whether children's supposed familiarity with area models might lead them to feel more confident when solving problems with area models and to report less difficulty while using area models. Although children's confidence ratings for the circular area condition were nominally higher than the other conditions, we found no reliable differences in confidence and difficulty across visual model condition. We also do not have specific information about the familiarity of these visual models within this sample, limiting our ability to interpret the confidence and difficulty findings. For a full picture of how familiarity with each type of visual model may impact children's perceived confidence, difficulty, and performance, it may be necessary to assess the frequency with which children encounter these models across many years during elementary instruction, as the nature of children's experiences with visual models may change over time. For example, although circular area models are introduced in partitioning tasks very early in elementary school, rectangular area models are recommended (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and often used to represent multiplication concepts (Shin & Lee, 2018; Tsankova & Pjanic, 2009; Webel & DeLeeuw, 2016; Wu, 2011). Despite this limitation, it is still notable that children's confidence and difficulty ratings did not reflect the clear patterns of successful conceptual reasoning.

4.4. Educational implications and future directions

Both the IES Practice Guide for Fractions (Siegler et al., 2010) and CCSSM recommend the use of visual models to represent fraction operations, yet it was unclear whether number lines and area models would offer the same benefits to accuracy and conceptual understanding of fraction division. Even though we found a number line advantage in the current study, we are not recommending that teachers should avoid the use of area models in their classrooms when teaching children to reason about fraction division. First, as previously mentioned, additional research is needed to examine children's learning from direct instruction that includes visual models with different features. Second, representational fluency (Rau & Matthews, 2017) using multiple types of visual models that highlight complementary aspects of fraction concepts, is important, and there may be an optimal combination of visual models during instruction.

Future research is needed to investigate the types of classroom lessons that most effectively convey a deep and multi-faced conceptual understanding of fraction division that would support a transition to flexible symbol-only problem solving for advanced mathematics topics. This is especially true given that we did not see a lasting effect of visual model condition on far transfer problems (e.g., story generation) that did not include a researcher-generated visual model, suggesting that substantive learning may not have occurred during the problem-solving activity. It is unclear whether providing children with an explicit reminder (e.g., Gick & Holyoak, 1980) to draw a diagram to support their stories would have improved the likelihood of transfer, or whether simply asking children to solve more fraction division problems, possibly over the course of several sessions and with feedback, would have better prepared them to spontaneously transfer to new, conceptually-similar problems.

Another educational implication involved students' perceptions of problem difficulty and their confidence in solving the problems correctly. Children assigned to the circular area condition rated the fraction division problems as nominally less difficult than did children in the other visual model conditions. Also, children in the circular area condition were more confident that they had correctly solved the problems than were children in the other conditions. Though these mean differences were not significantly different, looking across our findings on accuracy, conceptual models, and difficulty, number lines may be "worth" their perceived difficulty, whereas circles may be unhelpful despite students' apparent comfort with them. In other words, our findings suggest that teachers should not be reluctant to use number lines even if their students appear to find them difficult. Further investigation is necessary to determine whether children's perceived comfort and familiarity with various visual models moderates their effectiveness. If so, it may be important to introduce more effective visual models, such as number lines, along with earlier fraction concepts to provide adequate time for familiarity with these representations to develop.

Finally, we believe these findings have important implications for pre-service and in-service teacher education. Generating conceptual models for fraction division is challenging for both children (Sidney & Alibali, 2015, 2017) and adults (Ball, 1990; Ma, 1999). Although the number line is a promising representation for supporting children's reasoning about the relative magnitude of two quantities, prior research on pre-service teachers' fraction reasoning demonstrates their discomfort and difficulty using linear models for representing fraction operations (e.g., Luo et al., 2011). For teachers to effectively incorporate number line activities into fraction division instruction, they must be proficient with these representations themselves. Thus, additional research is needed to more closely examine teacher and pre-service teacher learning with and about number line models for fraction operations.

4.5. Conclusions

The current study adds to a growing body of evidence that number lines facilitate children's accurate understanding of fraction magnitudes and their conceptual understanding of the relationships between fraction magnitudes as they solve fraction operation problems. Our results can inform the implementation of the CCSSM that recommends the use of visual models for representing fraction operations. Future work should investigate ways to promote children's use of visual models to help them reason about fraction division problems even when diagrams are not provided in given problems.

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